ON THE BIHARMONIC DIRICHLET PROBLEM: THE HIGHER DIMENSIONAL CASE

ATANAS STEFANOV, GREGORY C. VERCHOTA

ABSTRACT. We address the question for existence and uniqueness for the biharmonic equation on Lipschitz domains. In particular for the Dirichlet biharmonic problem on $D \subset \mathbb{R}^n$, we show solvability for data in L^p, $2- < p < 2(n-1)/(n-3)+$. This result complements known counterexamples due to Mazy'a-Nazarov-Plamenevskii and Verchota-Pipher, and is thus sharp at least in dimensions four and five.

1. INTRODUCTION

In this paper we study the problem for solvability of the biharmonic equation with L^p data on a Lipschitz domain $D \subset \mathbb{R}^n$, $n \geq 4$. For the Dirichlet problem

$$
(1) \qquad \left| \begin{array}{l} \Delta^2 u = 0 \quad \text{on} \quad D \\ u|_{\partial D} = f \quad \text{on} \quad \partial D \\ \dfrac{\partial u}{\partial N}|_{\partial D} = g \\ M(\nabla u) \in L^p(\partial D) \end{array} \right.
$$

we show that there exists an unique solution as long as $(f, g) \in L^p_1(\partial D) \times L^p(\partial D)$ and $2 - \varepsilon(D) < p < 2(n-1)/(n-3) + \varepsilon(D)$. This result settles in positive a question posed in [17] and should be viewed as a natural extension of the three dimensional results in [15].

For the Laplace's equation, which is the standard threshold for elliptic boundary value problems, the problem is far better understood. By the results in [6], [22]

$$
(2) \qquad (D_p) \left| \begin{array}{l} \Delta u = 0 \quad \text{on} \quad D \\ u|_{\partial D} = f \quad \text{on} \quad \partial D \\ M(u) \in L^p(\partial D) \end{array} \right.
$$

is uniquely solvable as long as $2 - \varepsilon(D) < p \leq \infty$, $f \in L^p(\partial D)$. Moreover, one has appropriate estimates of the solution in terms of the data

$$
\|M(u)\|_{L^p(\partial D)} \lesssim \|f\|_{L^p(\partial D)}.
$$

The important endpoint $p = \infty$ is included, because of the validity of the maximum principle. For the regularity problem,

$$
(3) \qquad (R_p) \left| \begin{array}{l} \Delta u = 0 \quad \text{on} \quad D \\ u|_{\partial D} = f \quad \text{on} \quad \partial D \\ M(\nabla u) \in L^p(\partial D) \end{array} \right.
$$

one has solvability and uniqueness provided $1 < p < 2 + \varepsilon$. Moreover,

$$
(4) \qquad \|M(\nabla u)\|_{L^p(\partial D)} \lesssim \|f\|_{L^p_1(\partial D)}.
$$

Even though $p = 1$ cannot be included in the range of (4) one has an appropriate replacement if we restrict the data f in the atomic Hardy spaces with one derivative $H^1_1(\partial D)$. Then

$$\|M(\nabla u)\|_{L^1(\partial D)} \lesssim \|f\|_{H^1_1(\partial D)}. \tag{5}$$

In a recent work [19], we have been able to extend (5) in *two dimensions* to the sharp estimate

$$\|M(\nabla u)\|_{L^{2/3}(\partial D)} \lesssim \|f\|_{H^{2/3}_1(\partial D)}.$$

The methods of [6] clearly showed the importance of having well-localized data such as Hardy spaces' atoms. Not surprisingly, one proves (4) by interpolating between (5) and the L^2 estimates of [22]. The results for the Dirichlet solvability can be obtained as a dual statements to the R_p results.

For higher order equations however, it was not immediately clear that solvability would follow from solvability for atomic data as it turned out in the harmonic case. In fact, for the regularity *biharmonic* problem (see Section 3 below) a counterexample due to Mazya-Nazarov-Plamenevskii shows that one cannot expect solvability for data in $L^p(\partial D)$, $p < 4/3$ if $D \subset \mathbb{R}^n$, $n \geq 5$. This in particular prevents estimates in L^1 for the solutions corresponding to atomic data. Subsequently, Pipher and Verchota [15] showed that for four dimensional domains, one cannot solve the regularity problem uniquely unless $p > 6/5$ and by the same token L^1 estimates for the solution corresponding to atomic data necessarily fail.

In dimension three however, Pipher and Verchota [15], basically carried out the atomic approach. They have showed L^1 appriori estimates for solution of the biharmonic regularity problem with atomic data. As a consequence, they were able to obtain existence and uniqueness for the Dirichlet problem, together with the estimates

$$\|M(\nabla u)\|_{L^p(\partial D)} \lesssim \|f\|_{L^p_1(\partial D)} + \|g\|_{L^p(\partial D)}.$$

for $2- < p < \infty$. In [16], by using the atomic estimates for the Green's function they have shown a maximum principle, together with a solvability in the Lipschitz class C^ε. In [17], the above techniques have been further developed to show L^2 solvability for higher order elliptic operators and an appropriate maximum principle in the *three dimensional* case.

Dahlberg and Kenig [5] used similar approach for the related three dimensional Lamé system to show maximum principle and C^ε solvability.

For the stationary Stokes system, Z. Shen ([18]) have shown maximum principle in the *three dimensional* case and some Sobolev-Besov type estimates (with a derivative loss) in the higher dimensional case.

Evidently, there must be some obstacle to prove estimates in the higher dimensional case ($n \geq 4$). While there are counterexamples for the biharmonic equation showing such estimates must fail, one has neither proof nor a counterexample for the L^p solvability of the Lamé and the Stokes sytems in dimensions higher than three, when p is away from 2. On the other hand, one should point out that the L^2 theory for all of the problems mentioned above has been developed in *all dimensions* ([17], [4], [7]).

The purpose of this paper is to shed some light on how to obtain sharp L^p estimates, when p is away from 1. To this end, we still need to exploit the basic idea that whenever the data is compactly supported the solution and its derivatives somewhat decay (on average) away from the support. The underlying difficulty with this approach is that for general L^p data one does not have any sort of reasonable decomposition into atoms. That is why we measure the solution in a new familly of weighted L^p norms with weights acting on arbitrary scales to accomodate various profiles of the initial data. We develop the corresponding (real)

interpolation theory for these spaces, so that our L^p estimates follow from estimates in L^2 and H^1.

We believe that appropriate analogues of these spaces can be used to obtain L^p estimates for the other elliptic boundary value problems mentioned above. We hope to report on these questions in a later paper.

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The organization of the paper is as follows. First, we give some background material for harmonic functions. In Section 3 we outline the L^2 theory for the Dirichlet and regularity biharmonic problems. The main estimates are in Section 5 followed by the definition of the weighted L^p spaces and their real interpolation properties. We state and prove our main results in Section 8. Finally, we offer some conjectures for the open problems alluded to above.

2. Some preliminaries

For simplicity, throughout this paper we will restrict our attention to Lipschitz domains D above graphs, i.e. for a fixed Lipschitz function with *compact support* φ

$$D = \{(x,y) : y > \varphi(x)\} \subset \mathbb{R}^n.$$

It is clear that the general case of non compactly supported φ can be obtained with the usual approximation techniques. We will always consider ∂D as being equipped with the surface measure $d\sigma = \sqrt{1+|\nabla\varphi|^2}dx$. Denote the Lipschitz character of the domain D as $L = \|\nabla\varphi\|_\infty$. We will frequently use the non-tangential boundary cone $\Gamma(Q) \subset D$ associated to every point $Q \in \partial D$.

$$\Gamma(Q) = \{Y \in D : |Y - Q| \leq (1 + L/10)\mathrm{dist}(Y, \partial D)\}.$$

The non-tangential maximal function with respect to $\Gamma(Q)$ of a function $u : D \to \mathbb{C}$ is

$$M(u)(Q) = \sup_{X\in\Gamma(Q)} |u(x)|.$$

Define the tangent vectors and tangent derivatives by

$$\frac{\partial}{\partial T_j}F = \langle T_j, \nabla F\rangle = D_jF + \frac{\partial\varphi}{\partial x_j}D_nF.$$

Let ω_n be the surface area of $\mathbb{S}^{n-1}$. Then

$$G(x) = \frac{|x|^{2-n}}{(n-2)\omega_n} \quad (n > 2)$$

is the fundamental solution for the Laplace's equation in $\mathbb{R}^n$. Define also the single and double layer potentials $\mathcal{S}$ and $\mathcal{K}$ by

$$\begin{aligned} \mathcal{S}(f)(X) &= \text{p.v.} \int_{\partial D} G(X-Q)f(Q)d\sigma(Q), \quad x \in \mathbb{R}^n \setminus \partial D \\ \mathcal{K}(f)(X) &= \text{p.v.} \int_{\partial D} \frac{\partial G}{\partial N_Q}(X-Q)f(Q)d\sigma(Q), \quad x \in \mathbb{R}^n \setminus \partial D \end{aligned}$$

We also define the formal adjoint of $\mathcal{K}$

$$\mathcal{K}^*(f)(X) = \text{p.v.} \int\limits_{\partial D} \frac{\partial G}{\partial N_X}(X-Q) f(Q) d\sigma(Q).$$

From the L^p boundedness of the Cauchy integral on Lipschitz curves [3], we have for $1 < p < \infty$

$$\begin{aligned} &\|M(\mathcal{K}f)\|_{L^p(\partial D)} \lesssim \|f\|_{L^p(\partial D)}, \\ &\|M(\nabla S f)\|_{L^p(\partial D)} \lesssim \|f\|_{L^p(\partial D)} \end{aligned} \tag{6}$$

Based on (6), the usual density argument and the jump relations, one defines the singular integral operators K_+, K_-, S_+, S_-, acting on the boundary ∂D as

$$\begin{aligned} \mathcal{K}_+ f(Q) &= \lim_{X \to Q; X \in \Gamma(Q)} \mathcal{K} f(Q) = \frac{1}{2} f(Q) + \mathcal{K} f(Q) \qquad a.e. \\ \mathcal{K}_- f(Q) &= \lim_{X \to Q; X \in -\Gamma(Q)} \mathcal{K} f(Q) = -\frac{1}{2} f(Q) + \mathcal{K} f(Q) \qquad a.e. \\ \mathcal{S}_+ f(Q) &= \lim_{X \to Q; X \in \Gamma(Q)} \mathcal{S} f(Q) \\ \mathcal{S}_- f(Q) &= \lim_{X \to Q; X \in -\Gamma(Q)} \mathcal{S} f(Q) \end{aligned}$$

The following theorem is essentially a reformulation of the existence and uniqueness statements for the regularity and Dirichlet problems (see Theorem 2.4 in [15]).

Theorem 1. *(Dahlberg-Kenig, Verchota) There exists $\varepsilon = \varepsilon(L) > 0$, so that*

$$\begin{aligned} \mathcal{K}_+ &: L^p(\partial D) \to L^p(\partial D), \quad 2 - \varepsilon < p < \infty, \\ \mathcal{K}_-^* &: L^q(\partial D) \to L^q(\partial D), \quad 1 < q < 2 + \varepsilon \end{aligned}$$

are invertible mappings. Moreover,

$$\|M(\nabla \mathcal{S} f)\|_{L^q(\partial D)} \sim \left\|\mathcal{K}_-^* f\right\|_{L^q(\partial D)} \sim \|f\|_{L^q(\partial D)}. \tag{7}$$

At this point, one is tempted to say that $\nabla \mathcal{S} f$ is an invertible operator, which is heuristically the case in view of (7). To make this statement precise, recall that one usually uses the "preferred" direction X_n to obtain

$$\|M(\nabla \mathcal{S} f)\|_p \sim \|M(D_n S f)\|_p \qquad \text{(Stein's Lemma)}.$$

Theorem 2.7 in [15] gives invertibility of $D_n S_\pm$ on $L^2(\partial D)$. In particular $D_n \mathcal{S}_\pm$ is one-to-one map in $L^2(\partial D)$. By (7) and the density of $L^2(\partial D)$ in $L^p(\partial D)$, it follows that $D_n \mathcal{S}_\pm$ is one-to-one map in $L^p(\partial D)$ as well. Thus,

$$D_n \mathcal{S}_\pm : L^p(\partial D) \to L^p(\partial D), \quad 1 < p < 2 + \varepsilon$$

is an invertible operator.
We also remark, that the (small) numbers ε that will appear frequently in our discussion, will not be the same at every appearance (although one can surely take the smallest one that appears and state the theorems with it). That is why, we will sometimes enjoy the liberty to denote by $A-$ a number which is equal to $A - \varepsilon$ for some potentially small $\varepsilon > 0$.

3. L^2 THEORY

We consider the Dirichlet and regularity problems separately, partly due to the technical issues and ambiguities arising in the definition of the regularity problem. Ideally, the regularity problem would ask for a biharmonic function u with a prescribed $D_n u$ and some *second* derivative on the boundary ∂D. However, even formally one cannot define two derivatives on the boundary due to the smoothness restrictions on the function φ.

3.1. L^2 Dirichlet problem.

For the Dirichlet problem (1) we have

Theorem 2. *(Dahlberg-Kenig-Verchota,[4]) There exists $\varepsilon > 0$, such that (1) is uniquely solvable whenever $2-\varepsilon < p < 2+\varepsilon$ and $(f,g) \in (L^p_1(\partial D) \times L^p(\partial D))$. Moreover, one has*

$$\|M(\nabla u)\|_{L^p} \lesssim \|\nabla u|_{\partial D}\|_{L^p(\partial D)} \sim (\|f\|_{L^p_1(\partial D)} + \|g\|_{L^p(\partial D)}), \tag{8}$$

$$|\nabla u(X)| \lesssim \operatorname{dist}(X, \partial D)^{-(n-1)/p}. \tag{9}$$

Even though Theorem 2 is important in its own right, we would like to somehow relate the existence and uniqueness statement in it to the invertibility in $L^p(\partial D)$ of certain singular integral operator. We will then essentially follow the approach from Theorem 1, to reduce the question for solvability of the regularity problem in $L^{p'}$ to the invertibility of the adjoint operator in $L^{p'}(\partial D)$. The way we choose to set up the regularity problem will be largely dictated by our goal to have the invertibility of the adjoint operator essentially equivalent to the solvability for the regularity problem.

This program has been carried on in the n dimensional case in [15]. Let B be the fundamental solution of the bilaplacian, $\Delta^2 B = \delta(X)$. According to Section 2 in [15], one can represent the solution as

$$u(X) = \lim_{t\to 0} \int_D G(X-Y) \frac{\partial}{\partial Y_n} \mathcal{K} f(Y + t e_n) dY,$$

where e_n is the unit vector in the X_n direction. Then the divergence theorem yields

$$\begin{aligned} D_i u(X) \;=\; & \sum_{j,k=1}^{n} \int_{\partial D} (N^n_P D_k - N^k_P D_k) D_k D_i B(X-P) \mathcal{K}_+ f(P) dP + \\ & + \int_{\partial D} (N^j_P D_k - N^k_P D_j) D_n D_i B(X-P) dP + \\ & + \int_{\partial D}\int_{\partial D} N^j_Q D_k G(Q-P) f(Q) dQ dP \end{aligned} \tag{10}$$

for $X \in \mathbb{R}^n \setminus \partial D$.

Lemma 3.6 and the proof of Theorem 3.7 in [15] show that to prove solvability for the Dirichlet problem in L^p, it will suffice to show that

$$T : f \to D_n u|_{\partial D}$$

is invertible, where $D_n u|_{\partial D}$ is the singular integral operator in (10) corresponding to $i = n$. Since the invertibility of T on L^2 follows from the Rellich identities (see (3.4) in [15]), Theorem 2 for $p = 2$ follows.

Remark The extension to $2-\varepsilon < p < 2+\varepsilon$ is automatic due to the Calderon's method [2]. In fact, it has been recently been shown in [9] (in a much more general situation) that

the set $\{p|T : L^p(\partial D) \to L^p(\partial D) \quad \text{is invertible}\}$ must be open. In particular, since we have verified that 2 is in the set, one has invertibility on a whole interval $2-\varepsilon < p < 2+\varepsilon$.

3.2. L^2 **regularity problem.** Throughout the paper, we will consider the regularity problem

$$(R_p)\left|\begin{array}{cc} \Delta^2 u & = 0 \\ D_n u|_{\partial D} & = f \\ \sum_{j=1}^{n-1}\langle \nabla_{T_j}, \nabla D_j u\rangle|_{\partial D} & = g \\ \|M(\nabla^2 u)\|_{L^p(\partial D)} & < \infty \end{array}\right.$$

The following theorem states that this particular version of the regularity problem has unique solution, obeying the usual estimates away from the boundary.

Theorem 3. *(Pipher-Verchota, [15]) There exists $\varepsilon = \varepsilon(L) > 0$, so that R_p is uniquely (up to a linear function) solvable for $(f, g) \in (L^p_1(\partial D) \times L^p(\partial D))$ and $2-\varepsilon < p < 2+\varepsilon$. In addition, there are the estimates*

- $|\nabla\nabla u(X)| \lesssim \operatorname{dist}(X, \partial D)^{-(n-1)/p}$,
- $\|M(\nabla\nabla u)\|_{L^p\partial D)} \lesssim \sum_j (\|\nabla_{T_j} f\|_{L^p(\partial D)} + \|g\|_{L^p(\partial D)})$.

4. Solvability of R_p is equivalent to the invertibility of T^*

In this section, we actually prove the equivalence of the R_p solvability $1 < p < 2$ and the invertibility of T^* on $L^p(\partial D)$. We remark that although these results hold true for the full range $1 < p < 2$, we will really need them only for the range $2(n-1)/(n+1)- < p < 2$.

4.1. **Invertibility of T^* implies solvability of R_p.** Let h be a harmonic function, such that $M(\nabla h) \in L^2(\partial D)$. Note that $|\nabla h(X)| \lesssim \operatorname{dist}(X, \partial D)^{(1-n)/2}$ and
$|\nabla^2 h(X)| \lesssim \operatorname{dist}(X, \partial D)^{(-1-n)/2}$ as $\operatorname{dist}(X, \partial D) \to \infty$. Take $t_0 > 2\max_{\mathbb{R}^{n-1}} \varphi(x)$ and $x_0 \in \mathbb{R}^{n-1}$. Following [15], one defines a primitive function H of h by

$$H(x,t) = \int_{t_0}^{t} h(x,s)ds - \int_{t_0}^{\infty}(h(x,s) - h(x_0,s))ds. \tag{11}$$

Based on the properties of h, it is not difficult to check that for $n \geq 4$, the function H is well defined and $|\nabla^2 H(X)| \lesssim \operatorname{dist}(X, \partial D)^{(1-n)/2}$ and $M(\nabla^2 H) \in L^2(\partial D)$. For data $(g_0, g_1) \in (L^2_1(\partial D) \times L^2(\partial D)$, take $h = \mathcal{K}(g_0) - \mathcal{S}(g_1)$. Fix $X^0 = (x_0, s_0) \notin D$, where x_0 was chosen before. Let $f \in L^2(\partial D)$. With H defined by (11) set

$$\begin{aligned} u(X) &= H(X) + \frac{1}{(n-2)\omega_n}\int_D \left(\frac{1}{|X-Y|^{n-2}} - \frac{1}{|X^0-Y|^{n-2}}\right)\frac{\partial}{\partial Y_n}\mathcal{S}f(Y)dY \\ &= H(X) - G(D_n\mathcal{S}f)(X). \end{aligned} \tag{12}$$

One can check (cf. Section 4, [15]) that u is a solution to R_2 as long as one can select $f \in L^2(\partial D)$ so that

$$T^* f = \frac{\partial}{\partial N_-} D_n G(D_n\mathcal{S}f) = \frac{\partial}{\partial N}\mathcal{K}g_0 - \frac{\partial}{\partial N}\mathcal{S}_- g_1 \in L^2(\partial D) \tag{13}$$

Therefore, the invertibility of T^* on L^2 implies solvability for R_2. One obtains similar statement for any $p : 1 < p < 2$, i.e. if T^* is invertible on $L^p(\partial D)$, then R_p can be solved uniquely.

4.2. **Solvability of R_p implies invertibility of T^*.** We show that solvability of R_p implies the invertibility of the operator $T^* : L^p(\partial D) \to L^p(dD)$, thus making these two statements equivalent. The argument is essentially a reprise of the proof of Lemma 6.2 in [15].

Suppose R_p is solvable for some p in the sense of Theorem 3. Consider the "reduced" regularity problem with zero $D_n u$ data

$$\left| \begin{array}{l} \Delta^2 u = 0 \quad \text{on} \quad D \\ D_n u|_{\partial D} = 0 \quad \text{on} \quad \partial D \\ \sum_{j=1}^{n-1} \langle \nabla_{T_j}, \nabla D_j u \rangle|_{\partial D} = a \quad \text{on} \quad \partial D \\ M(\nabla^2 u) \in L^p(\partial D) \end{array} \right. \tag{14}$$

where $a \in L^2(\partial D) \bigcap L^p(\partial D)$, but we will use only $\|a\|_{L^p(\partial D)}$ in our estimates. Then $T^{*-1}a$ is well defined. Define the harmonic function $h = -\mathcal{S}(a)$ and H by (11). Note also that according to (12) and (13) the solution can be written as

$$u_a = H - G(D_n \mathcal{S}(T^{*-1}a)).$$

In particular it follows that $\Delta u_a = D_n \mathcal{S}(T^{*-1}a)$. Hence, by Theorem 1, Stein's lemma and the assumed R_p solvability, one has

$$\begin{aligned} \left\| T^{*-1}a \right\|_p &\lesssim \left\| \frac{\partial}{\partial N} \mathcal{S}_+(T^{*-1}a) \right\|_p \lesssim \left\| M(\nabla \mathcal{S}(T^{*-1}a) \right\|_p \lesssim \\ &\lesssim \left\| M(D_n \mathcal{S}(T^{*-1}a) \right\|_p = \| M(\Delta u_a) \|_p \lesssim \|a\|_{L^p(\partial D)}. \end{aligned}$$

Since this inequality holds on a dense set of $L^p(\partial D)$, one has invertibility of T^* on $L^p(\partial D)$. Observe that, we have not used our full assumption for solvability of R_p, but only the solvability of (14). We will show that the regularity problem (14) has unique (up to a linear term) solution in the sequel.

4.3. **Construction of the solution for R_p.** We have shown that the invertibility of T^* on $L^p(\partial D)$ implies solvability for R_p. We show now how to obtain the solutions, if we know how to solve the "reduced" regularity problem (14). Heuristically, the solution to the "full" regularity problem is performed by solving a *harmonic* Dirichlet problem and a "reduced" regularity problem.

We consider the "full" regularity problem

$$\left| \begin{array}{l} \Delta^2 u = 0 \quad \text{on} \quad D \\ D_n u|_{\partial D} = f \quad \text{on} \quad \partial D \\ \sum_{j=1}^{n-1} \langle \nabla_{T_j}, \nabla D_j u \rangle|_{\partial D} = g \quad \text{on} \quad \partial D \\ M(\nabla^2 u) \in L^p(\partial D). \end{array} \right. \tag{15}$$

First, define h to be the unique harmonic function with Dirichlet data f and

$$\|M(\nabla h)\|_{L^p(\partial D)} \lesssim \|f\|_{L^p_1(\partial D)}.$$

Define a primitive H as in (11). We have

$$\left\| M(\nabla^2 H) \right\|_{L^p(\partial D)} \lesssim \|f\|_{L^p_1(\partial D)}.$$

and $\dfrac{\partial h}{\partial N} \in L^p(\partial D)$. Consider now the "reduced" regularity problem with data $a = g - \dfrac{\partial h}{\partial N} \in L^p(\partial D)$. Call the solution $\tilde{u}$. Then $u = H + \tilde{u}$ satisfy (15) with $M(\nabla^2 u) \in L^p(\partial D)$.

5. Main estimates

Our main results for the Dirichlet and regularity biharmonic problems are the following.

Theorem 4. *Let $D \subset \mathbb{R}^n$ be a Lipschitz domain. Then there is $\varepsilon = \varepsilon(D) > 0$, so that the biharmonic Dirichlet problem* (1) *has an unique solution for $2-\varepsilon < p < 2(n-1)/(n-3)+\varepsilon$. Moreover*

- $\|M(\nabla u)\|_{L^p(\partial D)} \lesssim \|f\|_{L^p_1(\partial D)} + \|g\|_{L^p(\partial D)}$,
- $|\nabla u(X)| \lesssim \mathrm{dist}(X, \partial D)^{(1-n)/p}$.

Theorem 5. *Let $D \subset \mathbb{R}^n$ be a Lipschitz domain. Then there is $\varepsilon = \varepsilon(D) > 0$, so that the biharmonic regularity problem* (15) *has an unique (up to a linear term) solution for $2(n-1)/(n+1) - \varepsilon < p < 2 + \varepsilon$. Moreover*

- $\|M(\nabla^2 u)\|_{L^p(\partial D)} \lesssim \|f\|_{L^p_1(\partial D)} + \|g\|_{L^p(\partial D)}$,
- $|\nabla^2 u(X)| \lesssim \mathrm{dist}(X, \partial D)^{(1-n)/p}$.

The uniqueness statements will be proved in Section 8. Let us only remark that the methods are standard and can actually be reconstructed from [15]. We note also that the restrictions for p come in naturally in the uniqueness results. Albeit not a proof of sharpness of our existence results in dimensions higher than five, this of course gives us some indication that might be the case.

As it was pointed out already, the existence statement for the Dirichlet problem would follow from the invertibility of T, when considered as an operator acting on $L^p(\partial D)$ for $2-\varepsilon < p < 2(n-1)/(n-3)+$. Similarly, the regularity problem can be solved based on the invertibility of $T^* : L^p(\partial D) \to L^p(\partial D)$, for $2(n-1)/(n+1)- < p < 2+$. Moreover, T is a bounded operator on $L^p(\partial D)$ from [3]. Thus, it will suffice to show that T^* is invertible in the range $2(n-1)/(n+1)- < p < 2+$. However that was a consequence of the solvability of the "reduced" regularity problem (14) in the same range. Thus, we aim at solving the "reduced" regularity problem in the range $2(n-1)/(n+1)- < p < 2+$. Observe that in the three dimensional case, the lower bound for p is $1-$, i.e. one needs to show that the reduced regularity problem has solution, when the data is in $H^1(\partial D)$ (it is quite standard in these type of problems to avoid L^1 and consider instead H^1). Pipher and Verchota have effectively used a Cacciopoli type argument to show indeed that such solutions exist and to prove the estimates on $\|M(\nabla^2 u)\|_{L^1(\partial D)}$. That was the content of the Main Lemma [15], p. 941. We have

Lemma 1. *Let $D \subset \mathbb{R}^n$, $n \geq 4$ be a Lipschitz domain above graph. Let a be a function supported in the unit ball of ∂D, $a \in L^2(\partial D)$. Then the unique L^2 solution u of the "reduced" regularity problem with data a satisfies*

$$\int_{\{(x,\varphi(x)):|x|\sim 2^j\}} M(\nabla^2 u)^2 \lesssim 2^{(-2-\varepsilon)j} \|a\|^2_{L^2(\partial D)}, \tag{16}$$

for some positive $\varepsilon = \varepsilon(D)$.

Remark Due to the lack of enough decay in the Green's function associated with low dimensions, Lemma 1 seems to be more complicated for $D \subset \mathbf{R}^4$. We will perform an additional argument in Section 9 to show that Lemma 1 holds in that case as well.

We assume $n \geq 5$. Before we go on to the proof, we will need some technical results. One has the following Fatou type theorem for biharmonic functions in Lipschitz domains. The version below is taken (with small changes) from Theorem 3.9 in [15].

Lemma 2. *Let $D \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Suppose that $\Delta^2 u = 0$ in D and $M(\nabla u) \in L^p(\partial D)$ for some $1 \leq p < \infty$. Then $u, \nabla u$ have non-tangential limits a.e. on D and $M(u) \in L^p(\partial D)$.*

The next lemma is a Cacciopoli type inequality for biharmonic functions on Lipschitz domains. It appears as Lemma 5.6 in [15].

Lemma 3. *Let $D \subset \mathbb{R}^n$ be a domain above Lipschitz graph. Let $\Omega_1 \subset \Omega_2 \subset D$ be bounded Lipschitz domains and $\Delta^2 u = 0$ in D with $M(\nabla^2 u) \in L^2(\partial D)$. Let also $1 < p < \infty$ and $d = \operatorname{dist}(\Omega_1, D \setminus \Omega_2)$. Then there is a constant C, depending only on the Lipschitz constant and p, so that*

$$\begin{aligned}\int\limits_{\Omega_1} |\nabla^2 u|^2 dX &\lesssim \|\nabla u\|_{L^{p'}(\partial D \cap \partial\Omega_2)} \left\|M(\nabla^2 u)\right\|_{L^p(\partial D)} + \\ &+ d^{-1}\|u\|_{L^{p'}(\partial D \cap \partial\Omega_2)} \left\|M(\nabla^2 u)\right\|_{L^p(\partial D)} + \\ &+ d^{-1}\|\nabla u\|_{L^2(\Omega_2)} \left\|\nabla^2 u\right\|_{L^2(\Omega_2)} + d^{-2}\|u\|_{L^2(\Omega_2)} \left\|\nabla^2 u\right\|_{L^2(\Omega_2)}.\end{aligned}$$

The next lemma is a somewhat more sophisticated variant of the usual hiding technique.

Lemma 4. *(Hiding lemma) Let $\{b_k\}$ be a sequence of positive numbers, with at most exponential rate of growth: $b_k \leq A 2^{Nk}$. Assume also for some integer l and $\varepsilon > 0$*

$$b_k^2 \leq B 2^{-k\varepsilon}((b_{k-l} + \ldots + b_{k+l})^{3/2} + 1).$$

Then there exists $\varepsilon' > 0$ and a constant C depending on A, B, l, ε, N, so that

$$b_k^2 \leq C B 2^{-k\varepsilon'}.$$

The proof of the lemma is elementary, so we omit the details.

Proof. (Lemma 1)

Let u be the unique L^2 solution to the (14) guaranteed by the L^2 regularity theory. Since, $M(\nabla^2 u) \in L^p(\partial D)$, we conclude by Lemma 2 that $u, \nabla u, \nabla^2 u$ have non-tangential limits and the maximal functions taken over some cones with finite height are in $L^p(\partial D)$. Define

$$\tilde{u}(x) = \frac{1}{(n-3)\omega_{n-1}} \int\limits_{\mathbb{R}^{n-1}} \frac{a(y, \varphi(y))}{|x-y|^{n-3}} dy. \tag{17}$$

Since $\tilde{u}$ is a convolution of $a(y, \varphi(y))$ with the Green's function, we get $\Delta_x \tilde{u} = a$. Also, by differentiating the integral above, one obtains the estimates

$$|\tilde{u}(x)| \lesssim |x|^{3-n} \|a\|_2, \tag{18}$$

$$|\nabla_T \tilde{u}(x)| \lesssim |x|^{2-n} \|a\|_2, \tag{19}$$

$$|\nabla_T^2 \tilde{u}(x)| \lesssim |x|^{1-n} \|a\|_2, \tag{20}$$

for large x. Note that since $D_n u|_{\partial D} = 0$, one has

$$D_j u(x, \varphi(x)) = \frac{\partial u}{\partial x_j}(x, \varphi(x)). \tag{21}$$

By Lemma 2, one justifies the following calculation. Take a test function $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^{n-1})$ and perform two integration by parts to get

$$\int_{\mathbb{R}^{n-1}} \Delta\psi(x) u(x,\varphi(x))dx = -\sum_{j=1}^{n-1} \int_{\mathbb{R}^{n-1}} \frac{\partial \psi}{\partial x_j} D_j u(x,\varphi(x))dx =$$
$$= \sum_{j=1}^{n-1} \int_{\mathbb{R}^{n-1}} \psi(x) \left(\sum_{j=1}^{n-1} \langle \nabla_{T_j}, \nabla D_j u\rangle|_{\partial D} \right) dx = \langle \psi, a\rangle,$$

where in the second to the last identity above, we have used (21).

Hence, $u(x,\varphi(x))$ is a weak solution to the equation $\Delta_x v(x) = a(x,\varphi(x))$. But so is $\tilde{u}(x)$. Thus, by the Weyl's theorem for uniqueness of harmonic functions, (20) and $M(\nabla^2 u) \in L^2(dD)$, we deduce that $u(x,\varphi(x)$ and $\tilde{u}(x)$ differ by at most a linear term. By subtracting the linear term, we can assume that in fact $u(x,\varphi(x)) = \tilde{u}(x)$.

We now turn to the proof of (16). Note that the case of $j < 3$ follows from the L^2 regularity theory. Assume $j \geq 3$. We then dispose of the supremum in the definition of $M(\nabla^2 u)$ taken over the points inside D that are far away from ∂D. More specifically, take Γ_0 to be a cone pointing upward with vertex at the origin and with a large slope, say $100\|\varphi'\|_\infty$. If $Q \in \partial D$ then the intersection $\Gamma_0 \bigcap \Gamma(Q)$ consists of points X with $\mathrm{dist}(X,\partial D) \gtrsim |Q|$. Define

$$\begin{aligned} M_1(\nabla^2 u)(Q) &= \sup_{X\in\Gamma_0\bigcap\Gamma(Q)} |\nabla^2 u(X)| \\ M_2(\nabla^2 u)(Q) &= \sup_{X\in\Gamma(Q)\setminus\Gamma_0} |\nabla^2 u(X)|. \end{aligned}$$

We will show that

$$M_1(\nabla^2 u)(Q) \lesssim |Q|^{-1-(n-1)/(2-\varepsilon)}\|a\|_2 \qquad \text{for large} \quad Q, \tag{22}$$

which implies (16), when M is replaced with M_1. To show the pointwise estimate (22), we have by the L^2 Dirichlet theory, (17) and fractional integration

$$\|M(\nabla u)\|_{L^{2-\varepsilon}(\partial D)} \lesssim \|\nabla\tilde{u}\|_{L^{2-\varepsilon}(\partial D)} \lesssim \|a\|_{L^{2n/(n+2)-}(\partial D)} \lesssim \|a\|_{L^2(\partial D)}$$

and therefore

$$|\nabla u(X)| \lesssim \|a\|_2 \mathrm{dist}(X,\partial D)^{-(n-1)/(2-\varepsilon)}. \tag{23}$$

Consequently, for all $X \in \Gamma_0 \bigcap \Gamma(Q)$, we get by (23) and interior estimates

$$|\nabla^2 u(X)| \lesssim \|a\|_2 \mathrm{dist}(X,\partial D)^{-1-(n-1)/(2-\varepsilon)},$$

and hence since $\mathrm{dist}(X,\partial D) \gtrsim |Q|$

$$M_1(\nabla^2 u)(Q) \lesssim \|a\|_2 |Q|^{-1-(n-1)/(2-\varepsilon)}.$$

It remains to show the bound

$$\int_{\{(x,\varphi(x)):|x|\sim R\}} M_2(\nabla^2 u)^2 \lesssim \|a\|_2^2 R^{(-2-\varepsilon)}, \tag{24}$$

for $R \geq 10$ and some positive $\varepsilon > 0$.

For $1 \leq \tau \leq 2$, define the Carleson region Ω_τ^R above $Z_R = \{(x,\varphi(x)) : |x| \sim R\}$ as

$$\Omega_\tau = \Omega_\tau^R = \{(x,t) : R/\tau \leq |x| \leq R\tau, \varphi(x) < t < 100\tau R\|\varphi'\|_\infty\}.$$

From the L^2 regularity result on Ω_τ^R, we have

$$\int\limits_{\partial\Omega_\tau\cap\partial D} M_2(\nabla\nabla u)^2 \lesssim \int\limits_{\partial\Omega_\tau\setminus\partial D} |\nabla\nabla u|^2 + \sum_{j=1}^{n-1}\int\limits_{\partial\Omega_\tau\cap\partial D} |\nabla_{T_j}\nabla u|^2. \tag{25}$$

By (20), we get for $Q=(x,\varphi(x))\in\partial\Omega_\tau\cap\partial D$

$$|\nabla_{T_j}\nabla u(Q)| \lesssim \int\limits_{\mathbb{R}^{n-1}} \frac{|a(y,\varphi(y))|}{|x-y|^{n-1}}dy \lesssim \frac{1}{R^{n-1}}\|a\|_2,$$

since $|x|\sim R$. Hence

$$\sum_{j=1}^{n-1}\int\limits_{\partial\Omega_\tau\cap\partial D} |\nabla_{T_j}\nabla u|^2 \lesssim \frac{\|a\|_2^2}{R^{n-1}}.$$

Averaging (25) in $\tau\in(1,2)$ (see p. 944, [15]) yields

$$\int\limits_{\partial\Omega_1\cap\partial D} M_2(\nabla\nabla u)^2 \lesssim R^{-1}\int\limits_{\Omega_2} |\nabla\nabla u|^2 + \frac{\|a\|_2^2}{R^{n-1}}. \tag{26}$$

By Lemma 3 with $p=2-\varepsilon$, we can choose a domain Ω_3, such that $\Omega_2\subset\Omega_3\subset D$, $\mathrm{dist}(\Omega_2, D\setminus\Omega_3)\sim R$ and

$$\begin{aligned}\int\limits_{\Omega_2}|\nabla^2 u|^2 &\lesssim \|\nabla u\|_{L^{2+}(\partial D\cap\partial\Omega_3)}\big\|M(\nabla^2 u)\big\|_{L^{2-}(\partial D)} + \\ &+ R^{-1}\|u\|_{L^{2+}(\partial D\cap\partial\Omega_3)}\big\|M(\nabla^2 u)\big\|_{L^{2-}(\partial D)} + \\ &+ R^{-1}\|\nabla u\|_{L^2(\Omega_3)}\big\|\nabla^2 u\big\|_{L^2(\Omega_3)} + R^{-2}\|u\|_{L^2(\Omega_3)}\big\|\nabla^2 u\big\|_{L^2(\Omega_3)}.\end{aligned} \tag{27}$$

By the L^2 regularity theory

$$\big\|M(\nabla^2 u)\big\|_{L^{2-}(\partial D)} \lesssim \|a\|_{L^{2-}(\partial D)} \lesssim \|a\|_{L^2(\partial D)}. \tag{28}$$

The boundary terms are estimated via (18), (19)

$$\|\nabla u\|_{L^{2+}(\partial D\cap\partial\Omega_3)} \lesssim \frac{\|a\|_2}{R^{(n-3+)/2}} \tag{29}$$

$$\|u\|_{L^{2+}(\partial D\cap\partial\Omega_3)} \lesssim \frac{\|a\|_2}{R^{(n-5+)/2}}. \tag{30}$$

The fundamental theorem of calculus and (18) yield

$$\|u\|_{L^2(\Omega_3)} \lesssim \frac{\|a\|_2}{R^{(n-6)/2}} + R\|\nabla u\|_{L^2(\Omega_3)}. \tag{31}$$

Putting together (27), (28), (29), (30), (31) yields

$$\begin{aligned}\int\limits_{\Omega_2}|\nabla\nabla u|^2 &\lesssim \frac{\|a\|_2^2}{R^{(n-3+)/2}} + \frac{\|a\|_2\|\nabla^2 u\|_{L^2(\Omega_3)}}{R^{(n-2)/2}} + \\ &+ R^{-1}\|\nabla u\|_{L^2(\Omega_3)}\big\|\nabla^2 u\big\|_{L^2(\Omega_3)}.\end{aligned} \tag{32}$$

For some terms, the trivial estimate

$$\left\|\nabla^2 u\right\|_{L^2(\Omega_3)} \lesssim R^{1/2}\left\|M(\nabla^2 u)\right\|_{L^2(\partial D)} \lesssim R^{1/2}\|a\|_2 \tag{33}$$

will do. For others, we also have

$$\int_{\Omega_3} |\nabla^2 u|^2 \lesssim R\left(\int_{\Omega_3} M_1(\nabla^2 u)^2 + \int_{\Omega_3} M_2(\nabla^2 u)^2\right),$$

since it is possible to have $\Omega_3 \bigcap(\Gamma_0 \cap \Gamma(Q)) \neq \emptyset$ for some $Q \in \partial D$. However, by (22) we easily bound the contribution from $M_1(\nabla^2 u)$

$$\int_{\Omega_3} M_1(\nabla^2 u)^2 \lesssim R^{-2-\varepsilon}\|a\|_2^2. \tag{34}$$

Combining (26), (32), (33) and (34) gives

$$\int_{\partial\Omega_1\cap\partial D} M_2(\nabla^2 u)^2 \lesssim \frac{\|a\|_2^2}{R^{2+}} + R^{-3/2}\|\nabla u\|_{L^2(\Omega_3)}\left\|M_2(\nabla^2 u)\right\|_{L^2(\partial\Omega_3\cap\partial D)}. \tag{35}$$

To estimate $\int_{\partial\Omega_1\cap\partial D} M_2(\nabla^2 u)^2$ based on (35), we shall need to apply the Hiding lemma and apply Sobolev embedding. First, choose $p = 2-\varepsilon$ and denote its conjugate exponent by $p' = 2+$. Note that

$$\frac{1}{p} + \frac{n}{p'} - \frac{n-2}{2} = \frac{3}{2} - .$$

This is easily checked by setting $f(p) = 1/p + n/p' - (n-2)/2$ and verifying that this is a monotonically increasing function in $(1,2)$ with $f(2) = 3/2$. By Hölder, L^2 Dirichlet theory and the Sobolev embedding $W_{2,2}(\mathbb{R}^n) \hookrightarrow W_{2n/(n-2),1}(\mathbb{R}^n)$ we obtain

$$\begin{aligned} \int_{\Omega_3} |\nabla u|^2 &\lesssim \left(\int_{\Omega_3} |\nabla u|^p\right)^{1/p} R^{n/p'}\left(\frac{1}{|\Omega_3|}\int_{\Omega_3} |\nabla u|^{p'}\right)^{1/p'} \lesssim \\ &\lesssim R^{1/p}R^{n/p'}\|M(\nabla u)\|_{L^p(\partial D)}\left(\frac{1}{|\Omega_3|}\int_{\Omega_3} |\nabla u|^{2n/(n-2)}\right)^{(n-2)/2n} \lesssim \\ &\lesssim R^{1/p+n/p'-(n-2)/2}\|a\|_2\left\|\nabla^2 u\right\|_{L^2(\Omega_3)} \lesssim R^{2-}\|a\|_2\left\|M_2(\nabla^2 u)\right\|_{L^2(\partial\Omega_3\cap\partial D)} \end{aligned}$$

Combining this last estimate with (35) yields

$$\int_{\partial\Omega_1\cap\partial D} M_2(\nabla^2 u)^2 \lesssim \frac{\|a\|_2^2}{R^{2+}} + R^{-1/2-}\|a\|_2^{1/2}\left\|M_2(\nabla^2 u)\right\|^{3/2}_{L^2(\partial\Omega_3\cap\partial D)}. \tag{36}$$

For $R = 2^k$, denote

$$b_k = \frac{2^k\|M_2(\nabla^2 u)\|_{L^2(\partial\Omega_1\cap\partial D)}}{\|a\|_2}$$

It is clear that

$$\frac{2^k\|M_2(\nabla^2 u)\|_{L^2(\partial\Omega_3\cap\partial D)}}{\|a\|_2} \lesssim \sum_{l=-10}^{10} b_{k+l}.$$

Thus we rewrite (36) as

$$b_k^2 \lesssim 2^{-k\varepsilon}(1 + (\sum_{l=-10}^{10} b_{k+l})^{3/2}).$$

An application of the Hiding Lemma (Lemma 4) to the sequence $\{b_k\}$ yields

$$b_k^2 \lesssim 2^{-k\varepsilon'}$$

or equivalently (24) for $R = 2^k$, whence the general case for non dyadic R follows immediately. $\square$

6. Some interpolation tools

This section will provide some background material on interpolation spaces. Although the facts are quite standard, the definitions of the spaces that we are about to use differ from one source to another. Since the interpolation formulas in the endpoint cases can not be easily put into an unified framework, many authors prefered to leave them behind. That is why, we felt we needed to present the basic theorems, with an emphasis on the ones which we will be using in the sequel. The reader might find it convenient to skip this section at first and use it only as a reference later, when need arise.

We start with the definition of the Triebel-Lizorkin spaces for which we follow the exposition in [21]. Let $\psi_0 \in \mathcal{S}_0^\infty(\mathbb{R}^1)$ with supp $\psi_0 \subset (-1,1)$ and $\psi_0(x) = 1$ for $-1/2 < x < 1/2$. Take $\psi(x) = \psi_0(2x) - \psi_0(x)$. Call $\psi_j(x) = \psi(2^{-j}x)$ for $j \geq 1$. Then

$$\sum_{j=0}^{\infty} \psi_j(x) = 1.$$

This defines the Littlewood-Paley projection operators

$$S_j f(x) == \int\limits_{\mathbb{R}^n} f(x-y)\widehat{\psi_j}(|y|)dy,$$

which essentially restrict the Fourier transform of f to the annulus $\{\xi : |\xi| \sim 2^j\}$. The Triebel-Lizorkin spaces $F^s_{p,q}$ are defined as as the set of all functions f with

$$\|f\|_{F^s_{p,q}} = \left\| \left\| f * \widehat{\psi_j} \right\|_{l^q_s} \right\|_{L^p(\mathbb{R}^n)} = \left(\int\limits_{\mathbb{R}}^{n} \left(\sum_{j=0}^{\infty} 2^{jsq} |S_j f(x)|^q \right)^{p/q} dx \right)^{1/p} < \infty.$$

For a background material on $F^s_{p,q}$ and their relation with Lebesgue and Besov spaces one may consult [21], p. 169.

Introduce the spaces $F^s_{p,q,(r)}$

$$F^s_{p,q,(r)} = \left\{ f : \left\| \|S_j f\|_{l^q} \right\|_{L^{p,r}(\mathbb{R}^n)} < \infty \right\},$$

which will used to describe $(F^{s_1}_{p_1,q_1}, F^{s_2}_{p_2,q_2})_{(\theta,p)}$. We remind that $L^{p,r}$ are the Lorentz spaces and $L^{p,p} \equiv L^p$, while $L^{p,\infty}$ is the weak L^p space. The following lemma is due to Triebel (cf. Theorem 1, p. 184, [21]).

Lemma 5. *Let $1 \le p_1 < \infty$, $1 < p_2, q < \infty$, $p_1 \neq p_2$. Let $1/p = (1-\theta)/p_1 + \theta/p_2$. Then*

$$(F^s_{p_1,q}, F^s_{p_2,q})_{(\theta,p)} = F^s_{p,q,(r)}$$

Remark The proof provided by Triebel in [21] does not explicitly state the case $p_1 = 1$, which we will need. Actually, one can easily follow the argument in [1] for Besov spaces (see p. 153, (2)), where the case $p_1 = 1$ is covered. Alternatively, close inspection of the proof in [21] shows that the argument goes through in the case $p_1 = 1$, $p_2 \neq 1$, if one uses the formula

$$(L^1, L^2)_{(\theta,1)} = L_{p,1}.$$

It is also a standard fact (see [8]) that the Hardy spaces can be put in the framework of Triebel-Lizorkin spaces, namely $H^1 = F^0_{1,2}$. By the Littlewood-Paley theorem $L^q = F^0_{q,2}$. By Lemma 5 one gets for $1 \le r < \infty$

$$(H^1, L^q)_{(\theta,r)} = F_{p,2,(r)} \qquad \text{where} \quad 1/p = (1-\theta) + \theta/q. \tag{37}$$

7. Interpolation space

In this section, we introduce a familly of function spaces $X^p_\sigma(\mathbb{R}^n)$ in which we will measure our solutions. Heuristically, σ is a weight index, while p stands for L^p integrability as in the usual Lebesgue spaces. We have several objectives. First, we would like an inherent connection with L^p spaces, i.e. we wish L^p to be somehow embedded into this familly of spaces. In fact, we will show that $X^p_0(\mathbb{R}^n) \hookrightarrow L^p$. Second, we would like this familly to have a good "scale" properties. While we cannot quite satisfy that with our construction, we still have an almost precise formula for interpolation of a pair of X^p_σ spaces with the real interpolation method. Lastly, we would like to be able to translate the estimates in Lemma 1 into estimates for the solutions in X^p_σ spaces.

To this end, fix a sequence $\{a_m\} \subset \mathbb{R}^n$, dense in the unit ball of $\mathbb{R}^n$ and $|a_m| \le 2$.

For the interpolation scheme, we will represent X^p_σ as a factor space of a sequence space and later we will use standard interpolation results to find the intermediate spaces. Let Y^p_σ be the sequence space consisting of functions $(f_{l,m})$, $f_{l,m} : \mathbb{R}^n \to \mathbb{C}$, such that

$$\|(f_{l,m})\|_{Y^p_\sigma(\mathbb{R}^n)} = \sum_{l \le 0, m \ge 0} \left\| \sum_{j \ge 0} f_{l,m}(x)\psi_j(2^{-l}|x - a_m|)2^{j\sigma} \right\|_{L^p(\mathbb{R}^n)} < \infty.$$

Here, we restrict our attention to $1 \le p \le 2$ and $\sigma \in J$, where J is a finite interval (we will use below $J = (-(n-3)/2+, 1+))$, although there is no principal difficulty to extend the interpolation theory of Y^p_σ spaces beyond that. Define

$$C = \left\{ (f_{l,m}) \in \bigcap_{1 \le p \le 2, \sigma \in J} Y^p_\sigma : \sum_{l \le 0, m \ge 0} f_{l,m} = 0 \right\}.$$

Note that $\bigcap_{1 \le p \le 2, \sigma \in J} Y^p_\sigma \neq \varnothing$, since at least the functions in the Schwartz class $\mathcal{S}$ have the required decay properties.

Claim 1. *C is a closed subset in each Y^p_σ ,$1 \le p \le 2, \sigma \in J$.*

Proof. Take a convergent in Y^p_σ sequence $(f^{(n)}_{l,m})$ in C. We need to show that the limit $(f_{l,m}) \in Y^p_\sigma$ adds up to zero. Fix $j \geq 1$. We have

$$\left\| \sum_{l\leq 0, m\geq 0} (f^{(n)}_{l,m} - f_{l,m})\chi_{|x-a_m|\leq 2^j} \right\|_{L^p} \lesssim 2^{-\sigma j} \left\| (f^{(n)}_{l,m}) - (f_{l,m}) \right\|_{Y^p_\sigma} \to 0 \quad \text{as} \quad n \to \infty.$$

Since $|a_m| \leq 2$ and $\sum f^{(n)}_{l,m} = 0$, we conclude

$$\sum f_{l,m}(x) = 0 \quad \text{for} \quad |x| < 2^j/2.$$

This implies Claim 1, since j was arbitrary. □

Define $X^p_\sigma(\mathbb{R}^n) = Y^p_\sigma(\mathbb{R}^n)/C$ or equivalently as the space of all functions $f : \mathbb{R}^n \to \mathcal{C}$ so that

$$f = \sum_{l\leq 0, m\geq 0} f_{l,m}; \qquad (f_{l,m}) \in Y^p_\sigma;$$

$$\|f\|_{X^p_\sigma(\mathbb{R}^n)} = \inf_{f=\sum f_{l,m}} \sum_{l\leq 0, m\geq 0} \left\| \sum_{j\geq 0} f_{l,m}(x)\psi_j(2^{-l}|x-a_m|)2^{j\sigma} \right\|_{L^p(\mathbb{R}^n)} < \infty.$$

Since ψ_j is a partition of unity, we observe that for $\sigma = 0$, we have

$$\|f\|_{X^p_0} = \inf_{f=\sum f_{l,m}} \sum_{i,l,m} \|f_{l,m}(x)\|_{L^p(\mathbb{R}^n)} \geq \|f\|_{L^p}.$$

Thus we obtain

$$X^p_0(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n). \tag{38}$$

Turning to the interpolation issues, we would really like for the complex interpolation formula

$$[X/C, Y/C]_\theta = [X, Y]_\theta/C$$

to hold. Although heuristically right, the formula might fail since C does not seem to be a complemented subspace in Y^p_σ (see the discussion on p. 120 in [21]) and as far as we know it is an open question whether the complementability condition is really necessary. However, there is the following real interpolation version, which does not require complementability of C. It is due to Petunin ([14], see also [21], p. 120).

Lemma 6. *(Petunin) Let (X_0, X_1) be an interpolation couple, and $C \subset X_0 \cap X_1$ is a closed subspace of both X_0 and X_1. Then for $0 < \theta < 1$, $1 \leq p < \infty$*

$$(X_0/C, X_1/C)_{(\theta,p)} = (X_0, X_1)_{(\theta,p)}/C.$$

We are now ready to state our main interpolation result.

Lemma 7. *Let $1 \leq p_1 < 2$, $1 < p_1 < 2$ and $0 < \theta < 1$. For any $\sigma_1, \sigma_2 \in J$, define $\sigma = (1-\theta)\sigma_1 + \theta\sigma_2 \in J$ and $1/p = (1-\theta)/p_1 + \theta/p_2$. Then*

$$(X^{p_1}_{\sigma_1}, X^{p_2}_{\sigma_2})_{(\theta,1)} \hookrightarrow X^p_\sigma. \tag{39}$$

In other words, the indices θ, p in X^p_σ interpolate like the usual weights and integrability indices for weighted L^p spaces.

Proof. By Petunin's Lemma since C is closed in both $Y^{p_1}_{\sigma_1}$ and $Y^{p_2}_{\sigma_2}$, it will suffice to show that

$$(Y^{p_1}_{\sigma_1}, Y^{p_2}_{\sigma_2})_{(\theta,1)} \hookrightarrow Y^p_\sigma \tag{40}$$

under the same restrictions on p_1, p_2 and θ. Write

$$w^\sigma_{m,l}(x) = \sum_j \psi_j(2^{-l}|x-a_m|)2^{j\sigma p}.$$

Then

$$Y^p_\sigma = l^1_{m,l}(L^p(w^\sigma_{m,l}(x)dx)),$$

where we use the notation l^1_i for the sequence space l^1 indexed by i. Standard interpolation results stipulate that the real interpolation method applied to spaces in the form $l^1(A_j)$ results in spaces in the form $l^1(A)$, where $A = (A_1, A_2)_{(\theta,1)}$. In our situation, by the theorem on p. 121 in [21]

$$\begin{aligned}(Y^{p_1}_{\sigma_1}, Y^{p_2}_{\sigma_2})_{(\theta,1)} &= (l^1_{m,l}(L^{p_1}(w^{\sigma_1}_{m,l}(x)dx))), l^1_{m,l}(L^{p_2}(w^{\sigma_2}_{m,l}(x)dx)))_{(\theta,1)} = \\ = \quad & l^1_{m,l}(L^{p_1}(w^{\sigma_1}_{m,l}(x)dx)), L^{p_2}(w^{\sigma_2}_{m,l}(x)dx))_{(\theta,1)}).\end{aligned}$$

At this point, we would have prefered to have a complex interpolation method applied to our spaces, since weighted L^p spaces do not behave very well under real interpolation. Actually, there exist general formulas that describe the (real interpolation) intermediate spaces of weighted L^p spaces, but they are too complicated. Instead, we will pass to the complex interpolation functor by the well known relation (see for example p. 102 in [1])

$$(X, Y)_{(\theta,1)} \hookrightarrow [X, Y]_\theta.$$

Thus, since

$$[L^{p_1}(w^{\sigma_1}), L^{p_2}(w^{\sigma_2})]_\theta = L^p(w^{(1-\theta)\sigma_1+\theta\sigma_2}) = L^p(w^\sigma),$$

we obtain

$$(Y^{p_1}_{\sigma_1}, Y^{p_2}_{\sigma_2})_{(\theta,1)} \hookrightarrow Y^p_\sigma,$$

which establishes our interpolation step. □

8. Existence and uniqueness

We start this section with the existence statement in Theorem 4 and Theorem 5.

Proof. (Existence) As it was pointed out allready (see the discussion after Theorem 5), it suffices to show existence for the "reduced" regularity problem in the range $2(n-1)/(n+1)- < p < 2+$. As always things are reduced to showing the estimate

$$\left\| M(\nabla^2 u_a)\right\|_{L^p(\partial D)} \le C\|a\|_{L^p(\partial D)},$$

for smooth data a and a constant C which is independent of a. Set the sublinear operator $Ta(X) = M(\nabla^2 u_{a\psi_0(|\cdot|)})(X)$ for $X \in \partial D$, where ψ_0 is the fixed smooth cut-off of $(-1,1)$. In other words, we take the solution that corresponds to $a\psi_0(|\cdot|)$ instead of u_a. This is done in order to localize the problem to data supported in the unit ball. We will show that

$$\|Ta\|_{X^1_{-(n-3)/2+}(\partial D)} \le C\|a\|_{H^1(\partial D)}, \tag{41}$$

$$\|Ta\|_{X^2_{1+}(\partial D)} \le C\|a\|_{L^2(\partial D)}, \tag{42}$$

with the usual identification of functions on ∂D with functions on $\mathbb{R}^{n-1}$.

Thus by the definition of T, it will suffice to prove (42) for functions a with support in the unit ball. To this end, take $a_m = 0$ and $l = 0$, $f_{0,0} = M(\nabla^2 u_{\psi_0 a})$ in the definition of $\|M(\nabla^2 u_a)\|_{X^2_{1+}(\partial D)}$ and observe that (42) follows from the statement of Lemma 1.

For (41), let $\sigma = -(n-3)/2 + \varepsilon/4$, where $\varepsilon > 0$ is the positive number guaranteed by Lemma 1. We expand $a = \sum_m \lambda_m b_m$ in sums of H^1 atoms. Similar argument as the one above shows that it will suffice to consider $b_m \psi_0$ instead of b_m. We estimate first the contribution of atoms with diameter$(supp b_m) \geq 1/10$. Set

$$g = \sum_{m: size(supp b_m) \geq 1/10} \lambda_m b_m.$$

Clearly $g \in L^1 \cap L^\infty$. By Cauchy-Schwartz and Lemma 1

$$\left\|M(\nabla^2 u_{g\psi_0})\right\|_{X^1_\sigma} \lesssim \sum_{j\geq 0} 2^{j\sigma} \left\|M(\nabla^2 u_{g\psi_0})\right\|_{L^1(|x|\sim 2^j)} \lesssim \sum_{j\geq 0} 2^{j(\sigma+(n-3)/2-\varepsilon/2)} \|g\psi_0\|_{L^2} \lesssim \|g\|_{L^2}.$$

For the "small" support atoms, set $supp\ b_m = B(z_m, r_m)$. Let $l_m : r_m \sim 2^{l_m} < 1/10$. Choose $q = q(m)$ such that $|a_q - z_m| < r_m/10$. Thus, by the triangle inequality, we have

$$\left\|T(\sum \lambda_m b_m)\right\|_{X^1_\sigma} \lesssim \sum_m |\lambda_m| \sum_{j\geq 0} 2^{j\sigma} \left\|M(\nabla^2 u_{\psi_0 b_m})\right\|_{L^1(|x-a_{q(m)}|\sim 2^{l_m+j})}.$$

Thus, it will be enough to prove for a fixed "small" atom b_m

$$\sum_{j\geq 0} 2^{j\sigma} \left\|M(\nabla^2 u_{\psi_0 b_m})\right\|_{L^1(|x-a_{q(m)}|\sim 2^{l_m+j})} \lesssim 1. \tag{43}$$

We have by Cauchy-Schwartz and Lemma 1 (with the appropriate scaling)

$$\begin{aligned} &\sum_j 2^{j\sigma} \left\|M(\nabla^2 u_{\psi_0 b_m})\right\|_{L^1(|x-a_{q(m)}|\sim 2^{l_m+j})} \lesssim \\ \lesssim\ &\sum_j 2^{j\sigma} 2^{(l_m+j)((n-1)/2)} \left\|M(\nabla^2 u_{\psi_0 b_m})\right\|_{L^2(|x-a_{q(m)}|\sim 2^{l_m+j})} \lesssim \\ \lesssim\ &\sum_j 2^{j(\sigma+(n-3)/2-\varepsilon/2)} 2^{l_m(n-1)/2} \|\psi_0 b_m\|_{L^2} \lesssim 1, \end{aligned}$$

which establishes (43) and thus (41).

We show now that (41) and (42) imply

$$\|Ta\|_{L^p} \lesssim \|a\|_{L^p}, \tag{44}$$

for the range $2(n-1)/(n+1)- < p < 2+$.

Estimate (44) follows from the next lemma with $\sigma_1 = -(n-3)/2+$, $\sigma_2 = 1+$ and $\theta = (n-3)/(n-1)-$ for $p \sim 2(n-1)/(n+1)-$ and then by complex interpolation with the L^2 theory.

Lemma 8. *Let $\sigma_1 < 0$ and $\sigma_2 > 0$, such that $(1-\theta)\sigma_1 + \theta\sigma_2 = 0$ and $1/p = (1-\theta) + \theta/2$ and assume that for a sublinear operator T*

$$T : H^1 \to X^1_{\sigma_1},$$
$$T : L^2 \to X^2_{\sigma_2},$$

Then

$$T : L^{p+} \to L^{p+}.$$

Proof. By the real interpolation method, we get

$$T : (H^1, L^2)_{(\theta,1)} \to (X^1_{\sigma_1}, X^2_{\sigma_2})_{(\theta,1)}.$$

According to Lemma 7, one has

$$(X^1_{\sigma_1}, X^2_{\sigma_2})_{(\theta,1)} \hookrightarrow X^p_0 \hookrightarrow L^p.$$

Therefore,

$$\|Ta\|_{L^p} \lesssim \|Ta\|_{(X^1_{\sigma_1}, X^2_{\sigma_2})_{(\theta,1)}} \lesssim \|a\|_{(H^1,L^2)_{(\theta,1)}}.$$

By (37)

$$\|Ta\|_{L^p} \lesssim \|a\|_{F^0_{p,2,(1)}}. \tag{45}$$

From the L^2 estimate for T, we have in particular

$$\|Ta\|_{L^2} \lesssim \|a\|_{L^2} = \|a\|_{F^0_{2,2,(2)}} \tag{46}$$

The interpolation of $F^0_{p,2,(1)}$ spaces is in fact very similar to the interpolation for the usual Triebel-Lizorkin spaces. Set $\delta > 0$ and let $p_\delta : 1/p_\delta = (1-\delta)/p + \delta/2$. It is clear that $p < p_\delta < p + O(\delta)$. By an argument similar to those in [21], p. 185, claim (c), with the appropriate replacement of L^q with either $L^{q,1}$ or $L^{q,2}$, one gets

$$(F^0_{p,2,(1)}, F^0_{2,2,(2)})_{(\delta,p_\delta)} = F^0_{p_\delta,2,(p_\delta)} = F^0_{p_\delta,2} = L^{p_\delta},$$

where the last identity is the Littlewood-Paley theorem for L^{p_δ}. Also

$$(L^p, L^2)_{(\delta,p_\delta)} = L^{p_\delta,p_\delta} \equiv L^{p_\delta}.$$

Interpolation between (45) and (46) with (δ, p_δ) yields

$$\|Ta\|_{L^{p_\delta}} \lesssim \|a\|_{L^{p_\delta}}.$$

□

It remains to observe that for data a whose support is inside $\{x : |x| < 1/2\}$, we have $a\psi_0 \equiv a$ and therefore (44) reads

$$\left\|M(\nabla^2 u_a)\right\|_{L^p} \lesssim \|a\|_{L^p},$$

for $2(n-1)/(n+1)- < p < 2+$. But in this last estimate, one can rescale to prove

$$\left\|M(\nabla^2 u_a)\right\|_{L^p} \lesssim \|a\|_{L^p},$$

for data a having compact support. The usual approximation techniques finish the proof. □

For the uniqueness of the regularity problem, we refer the reader to Lemma 6.9 in [15]. Although the statements include only the case $D \subset \mathbb{R}^3$, one can check that the higher dimensional case follows as well. In fact, the proof is a lot easier, since we are not anymore in the endpoint case $L^1(\partial D)$, where the predual space is unavailable.

For the uniqueness of the Dirichlet problem, we follow Theorem 7.1 in [15]. Suppose $\Delta^2 u = 0$ $u|_{\partial D} = 0$, $\partial u/\partial N = 0$ with $M(\nabla u) \in L^{2(n-1)/(n-3)+}$. Denote by D the original domain, which is translated by one unit up. Define the domain

$$\tilde{D} = \left\{X : X/|X|^2 \in D\right\}$$

and

$$\tilde{u} = |X|^{4-n} u(X/|X|^2).$$

The function $\tilde{u}$ is biharmonic with zero Dirichlet data. To show $\tilde{u} = 0$, it suffices by the L^2 uniqueness results of [4] to show that $M(\nabla\tilde{u}) \in L^{2+}(\partial D)$. Fix a cone Γ with vertex at $(0, 1 + \varphi(0)) \in \partial D$. Define

$$\tilde{\Gamma} = \left\{ X : \frac{X}{|X|^2} \in \Gamma \right\}$$

Following the estimates in [15], we find that

$$\sup_{X \in \Gamma(Q) \cap \tilde{\Gamma}} |\nabla\tilde{u}(X)|(Q) \in L^p(\partial D), \qquad \text{for all} \quad p < \infty.$$

For the iterated Hardy-Littlewood maximal function $\mathcal{M}_2(f) = M(M(f))$, there are the estimates ([15])

$$\left\| \sup_{X \in \Gamma(Q) \setminus \tilde{\Gamma}} |\nabla\tilde{u}(X)|(Q) \right\|_{L^{2+}(\partial D)} \lesssim \int_{\partial D, |P| \geq 1} (\mathcal{M}_2(\nabla u)(P))^{2+} |P|^{-2-} dP \lesssim$$

$$\lesssim \left(\int_{\partial D} (\mathcal{M}_2(\nabla u)(P))^{2(n-1)/(n-3)+} \right)^{(n-3)/(n-1)} \left(\int_{|P|>1} \frac{1}{|P|^{n-1+}} dP \right)^{2/(n-1)} < \infty.$$

Thus, $\|M(\nabla\tilde{u})\|_{L^{2+}(\partial D)} < \infty$ and by the uniqueness result of [4], $\tilde{u} = 0$. The uniqueness part of Theorem 5 follows. Note that the proof that we have presented breaks down for exponents $p < 2(n-1)/(n-3)-$. That seems to indicate that $2(n-1)/(n-3)-$ is the sharp exponent in dimensions $n \geq 6$.

9. The four dimensional case

In the four dimensional case, one does not obtain Lemma 1 (at least not directly), due to the lack of enough decay of the Green's function. We need an additional argument.

In the proof of Lemma 1, one obtains an estimate for the maximal function away from the boundary

$$M_1(\nabla^2 u)(Q) \lesssim \|a\|_2 |Q|^{-1-(n-1)/(2-\varepsilon)},$$

which implies (16) for M_1. For M_2 an identical argument as in the proof of Lemma 1 with $n = 4$, yields

$$\int_{\{(x,\varphi(x)):|x|\sim 2^j\}} M_2(\nabla^2 u_a)^2 \lesssim 2^{(-3/2-\varepsilon)j} \|a\|^2_{L^2(\partial D)}. \tag{47}$$

Thus we have (47) for $M(\nabla^2 u_a)$ as well. Hence, following the proof of estimates (41) and (42), one establishes

$$T : H^1 \to X^1_{-3/4+},$$
$$T : L^2 \to X^2_{3/4+}.$$

where $Ta = M(\nabla^2 u_{a\psi_0})$. According to Lemma 8 that implies

$$T : L^{4/3-}(\partial D) \to L^{4/3-}(\partial D).$$

By rescaling, we get for the "reduced" regularity problem

$$\big\|M(\nabla^2 u_a)\big\|_{L^p(\partial D)} \lesssim \|a\|_{L^p(\partial D)} \qquad 4/3- < p < 2. \tag{48}$$

This enables us to go back to the proof of Lemma 1 and improve on our estimate (47). Indeed, in the derivation of (26), we have used the Cacciopoli estimates from Lemma 3 with $p = 2-$. However, we have at our disposition (48), so we choose $p = 4/3-$. A quick inspection of the proof shows (just as before), that one gets

$$\int_{\{(x,\varphi(x)):|x|\sim 2^j\}} M_2(\nabla^2 u)^2 \lesssim 2^{(-2-\varepsilon)j}\|a\|^2_{L^2(\partial D)}, \tag{49}$$

which is Lemma 1 for $n = 4$. From there on, the proof proceeds as in the higher dimensional case $n \geq 5$.

10. Some open problems

In this section, we list some open problems for boundary value equations (or systems) that are somewhat related to the biharmonic equation. As we have mentioned already in the introduction, the common feature in all of those are the maximum principles and Hölder solvability in *three dimensions*, while for dimensions higher than three, only the L^2 theory has been developed.

10.1. **The Lamé system.** Let $D \subset \mathbb{R}^n$ be a Lipschitz domain and $\vec{u} = (u_1, \dots, u_n)$. We consider the Dirichlet problem

$$(D_p)\left|\begin{array}{l} \Delta\vec{u} + \nabla \mathrm{div}\vec{u} = 0, \\ \vec{u}|_{\partial D} = \vec{f} \\ M(u) \in L^p(\partial D) \end{array}\right.$$

and the traction problem,

$$(T_p)\left|\begin{array}{l} \Delta\vec{u} + \nabla \mathrm{div}\vec{u} = 0, \\ (\nabla\vec{u} + \nabla\vec{u}^t)N|_{\partial D} = \vec{g} \\ M(\nabla u) \in L^p(\partial D) \end{array}\right.$$

We remark that the traction problem seems to be the right substitute for the regularity problem for the Lame system.

For $n = 3$, Dahlberg-Kenig [5] have proved (weak) maximum principle in full analogy with the harmonic case, by reducing to the traction problem. For the traction problem, they have succesfully used the atomic estimates method of [15], together with the appropriate Cacciopoli type inequalities.

For $n \geq 4$, there is no counterexample to a maximum principle, but the best solvability result might be for $2- < p < 2(n-1)/(n-3)+$, just as in the biharmonic case. Let us only remark that as is well-known the solution $\vec{u}$ must be a biharmonic vector.

10.2. **(Stationary) Stokes system.**

$$\left|\begin{array}{l} \Delta\vec{u} = \nabla p \\ \mathrm{div}\vec{u} = 0 \\ \vec{u}|_{\partial D} = \vec{f} \end{array}\right.$$

In dimension three, Z. Shen proved (weak) maximum principle for the Stokes system and some Sobolev-Besov type regularity results (with a derivative loss) in $n \geq 4$. A natural question that arises is about the L^p solvability for dimensions higher than three.

10.3. **The polyharmonic equation.** We will be highly schematic for the definiton of the polyharmonic equation. The reader is refered to [17] for an extensive treatment of these higher order boundary value problems. For suitable differential operators on the boundary $P_0, \dots, P_{m-1}$ (P_i is of order i)

$$\left| \begin{array}{l} \Delta^m u = 0 \\ P_0 u|_{\partial D} = f_0, \dots, P_{m-1} u|_{\partial D} = f_{m-1}, \\ M(\nabla^{m-1} u) \in L^p(\partial D). \end{array} \right.$$

In [17], Pipher-Verchota have shown that the usual L^2 theory for all n, m holds and a (weak) maximum principle for $n = 3$, just as in the harmonic case. The L^p results again made use of the atomic estimates that we have alluded to earlier, and we ask whether an analog of X^p_σ spaces might be helpful to study the L^p solvability in dimensions higher than three. In particular, a potentially sharp estimate in the form

$$\left\| M(\nabla^{m-1} u) \right\|_{L^p(\partial D)} \lesssim \| u|_{\partial D} \|_{L^p_{m-1}(\partial D)},$$

might hold for all $2- < p < 2(n-1)/(n-3)+$.

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ATANAS STEFANOV, LGRT 1342, DEPARTMENT OF MATHEMATICS AND STATITICS, UNIVERSITY OF MASSACHUSETTS, AMHERST, MA 01003, USA

E-mail address: `STEFANOV@MATH.UMASS.EDU`

GREGORY VERCHOTA, 215, CARNEGIE HALL, DEPARTMENT OF MATHEMATICS, SYRACUSE UNIVERSITY, SYRACUSE, NY 13088, USA